Formalism of six operations and derived algebraic stacks

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1 Introduction

The formalism of six operations was originally introduced by A.Grothendieck and his collaborators in the study of étale cohomology. It naturally leads to many well-known results in cohomology theory like duality and Lefschetz trace formula. This partially justifies the slogan that the formalism of six operations are enhanced cohomology theories. In this talk, I will introduce the formalism of six operations. I will explain the relation between it and some cohomology theories (Topological, coherent, l-adic). Moreover, I will talk about the application of it to a nice class of derived algebraic stacks. And show that this leads to some nontrivial results of algebraic (homotopy) K-theory for stacks.

2 Formalism of six operations

[Gal21] The six operations here refers to pullback-pushforward adjunct $f^* \dashv f_*$, exceptional adjunct $f_! \dashv f^!$, and tensor-hom adjunct $\otimes \dashv \underline{\mathrm{Hom}}$. In algebraic geometry, $f^*, f_*, f_!$ exist in the category of sheaves in abelian groups. For example, $f_!$ is the *pushforward with compact support* defined as

$$\Gamma(U, f_!(F)) = \{s \in \mathcal{F}(f^{-1}U) |\ s \text{ has compact support}\}$$

Note that there is a transformation $f_! \Rightarrow f_*$ induced by the inclusion of sections. However, $f_!$ doesn't always admits a right adjoint as pushforward of sheaves. To resolve it, we may pass to the derived category of sheaves (denoted by $D(Sh(X))$). Indeed, for some nice (e.g. locally compact Hausdorff) space, there exists $f^! : D(Sh(Y)) \to D(Sh(X))$ as the right adjoint of $Rf_!$.

Recall that if $f : X \to *$ is a map of schemes, $R^i f_*(\mathcal{F}) = H^i(X, \mathcal{F})$ where $R^i f_* : Sh(X) \to Sh(*) = \mathrm{Mod}(\mathbb{Z})$ is the derived functor of f_*. Similarly, $R^i f_!(\mathcal{F}) = H^i_c(X, \mathcal{F})$. For derived category of sheaves, the derived functor $\mathbf{R}f_* : D(Sh(X)) \to D(Ab)$ satisfies $H^i\mathbf{R}f_*(\mathcal{F}) = R^i f_*(\mathcal{F}) = H^i(X, \mathcal{F})$. So we may view the formalism of six operations as enhanced cohomology theory.

Now for greater generality, we assume that $C : Sm_B \to \{\text{closed tensor triangulated categories}\}$ (or symmetric monoidal presentable ∞-categories) is a functor that satisfies 'the formalism of six operations'. In this case we may make the following definition, for the structure map $p : X \to B$,

$$H^\bullet = p_* p^* \mathbf{1}, \qquad H^\bullet_c = p_! p^* \mathbf{1}$$

$$H_\bullet = p_! p^! \mathbf{1}, \qquad H^{BM}_\bullet = p_* p^! \mathbf{1}$$

In fact, $H^i(X, \mathcal{F}) = \underline{\mathrm{Hom}}_{C(X)}(\mathbf{1}, p_*(\mathcal{F})[i])$.

We didn't say anything about the definition of the formalism of six operations yet. The formalism of six operations is the six functors defined above together with a collection of axioms for these functors. There is no universal definition of the formalism of six operations since different authors might have different set of axioms. They may include base change, projection formula, relative purity, localization, homotopy invariance etc. On the other hand, these axioms are not minimal. In the sense that one may take a smaller set of axioms that underlies the formalism of six functors.

3 Grothendieck Duality

Assume that $f : X \to *$ is a smooth map. $H^i_c(X)^* \simeq \underline{\mathrm{Hom}}(f_! f^* \mathbf{1}[i], \mathbf{1}) \simeq \underline{\mathrm{Hom}}(\mathbf{1}, f_* f^! \mathbf{1}[-i])$

1. X is an orientable smooth manifold of dimension d. $f^!\mathbb{Q} = \omega_{X,Q}[d] = \mathbb{Q}[d]$ since X is orientable. So $H^i_c(X, \mathbb{Q})^* = \underline{\mathrm{Hom}}(\mathbf{1}, f_*\mathbb{Q}[d-i]) = H^{d-i}(X, \mathbb{Q})$ (Poincare Duality).
2. X is a proper smooth variety over k of dimension d. Then $f^!k \simeq \omega_X[d]$. $H^i(X, \mathcal{O}_X)^* = \underline{\mathrm{Hom}}(\mathbf{1}, f_*\omega_X[d-i]) = H^{d-i}(X, \omega_X)$ (Serre Duality).

4 Derived algebraic stacks

Convention 4.1.0. We use the convention of [Kh22], derived algebraic stacks refers to derived 1-Artin stacks in [GR].

We introduce 'derived algebraic stacks' for greater generality. One may replace 'derived algebraic stacks' by 'algebraic stacks' in the following. Schemes, algebraic spaces are examples for algebraic stacks.

Definition 4.1.1.

$$\mathrm{Sch}_{\mathrm{aff}} = \{\text{Derived affine schemes}\} \longleftrightarrow \{\textit{simplicial commutative rings}\}^{op}$$

Definition 4.1.2. A derived algebraic stack $\mathcal{X}$ is a functor $\mathrm{Sch}_{\mathrm{aff}}^{op} \to Spc$ such that

1. It satisfies étale descent.
2. The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.
3. It admits a smooth covering of affine schemes.

Roughly speaking, a *scalloped stack* in the sense of [Kh22] is a nice class of derived algebraic stacks such that it 'locally' looks like a quotient stack. Scalloped stacks encode the information of G-schemes and thus generalize the equivariant motivic homotopy theory.

We can construct the ∞-category of motivic spectra $SH(\mathcal{X})$ for a scalloped stack $\mathcal{X}$. Moreover, the assignment $\mathcal{X} \mapsto SH(\mathcal{X})$ satisfies the formalism of six operations.

The *cohomology spectrum* with coefficient $\mathcal{F} \in SH(\mathcal{X})$ is defined by

$$R\Gamma(\mathcal{X}, \mathcal{F}) = \mathrm{Maps}_{SH(\mathcal{X})}(1_{\mathcal{X}}, \mathcal{F})$$

This definition implicitly coincides with the definition of cohomology in section 2. For the precise meaning of $\mathcal{F}$, see [Kh22].

Proposition 4.1.3 (Étale excision) **.** [Kh22, prop 5.10] Let $f : \mathcal{X}' \to \mathcal{X}$ be a representable étale morphism which induces an isomorphism away from a quasi-compact open immersion $j : \mathcal{U} \to X$. Then th commutative square

$$\begin{array}{ccc} id & \longrightarrow & j_*j^* \\ \downarrow & & \downarrow \\ f_*f^* & \longrightarrow & g_*g^* \end{array}$$

is homotopy cartisian, where $g : f^{-1}(\mathcal{U}) \to \mathcal{X}$.

Proof. By localization, it suffices to prove the square is cartisian after applying i^* and j^*. This follows by the smooth projection formula. □

5 Algebraic K-theory

Recall that for a *Waldhausen category* $\mathcal{C}$, we may construct a collection of Waldhausen categories $S_n(C)$ for all $n \geq 0$. It turns out that these $S_n(\mathcal{C})$ fits together and thus form a simplicial Waldhausen category $S_\bullet(\mathcal{C})$. $wS_\bullet(\mathcal{C})$ is , in each level, the subcategory of weak equivalences. We call $K(\mathcal{C}) = \Omega|wS_\bullet(\mathcal{C})|$ the algebraic K-theory space of $\mathcal{C}$. Then $K_n(\mathcal{C}) = \pi_n(K(\mathcal{C}))$ if $n \geq 0$. We can turn this space into a connective spectra by iterating this $S_\bullet$ construction.

$K^B(R)$ is the nonconnective K-theory spectrum for a ring R. This is constructed from $K(R)$ using Bass construction. In fact $K \simeq \tau_{\geq 0}(K^B)$.

These notions still make sense when $\mathcal{C}$ is a stable ∞-category. So we may make the following definition,

Definition 5.1.1. [Kh20] For a derived algebraic stack $\mathcal{X}$, we define the Bass-Thomason-Trobaugh K-theory spectra

$$K^B(\mathcal{X}) = K^B(D_{\text{perf}}(\mathcal{X}))$$

where $D_{\text{perf}}(\mathcal{X})$ is the stable ∞-category of perfect complexes on $\mathcal{X}$.

Indeed, the assignment $\mathcal{X} \mapsto K^B(\mathcal{X})$ is a Nisnevich sheaf of spectra on the site of *scalloped derived stacks*. However, it is not homotopy-invariant in general.

Definition 5.1.2. (Homotopy invariant K-theory [Kh20]) For a scalloped derived stack $\mathcal{X}$, consider the presheaf

$$K^B : Sm_{\mathcal{X}}^{op} \to Spt$$

We define the homotopy invariant K-theory spectrum $KH(\mathcal{X}) \simeq \varinjlim_{[n]\in\Delta^{op}} K^B(\mathcal{X} \times \mathbf{A}^n)$. Equivalently, it is the global section of $L_{A^1}K^B$.

Remark 5.1.3. A fundamental result of homotopy invariant K-theory is that KH is representable in the motivic spectra, i.e., $KH(\mathcal{X}) \simeq R\Gamma(\mathcal{X}, KGL)$ for all scalloped stack $\mathcal{X}$.

Now we are in the position to state the theorem.

Theorem 5.1.4 [Kh22, Corollary G] **.** The presheaf of spectra $\mathcal{X} \to KH(\mathcal{X})$ satisfies cdh descent on the site of scalloped stacks.

Proof. Since KH is representable, it suffices to prove that $\mathcal{F} \to f_*f^*\mathcal{F}$ satisfies cdh descent. Since cdh topology is generated by Nisnevich squares and abstract blowup squares. We can check descent for these squares individually. The descent for Nisnevich square is proposition 4.1.3. It remains to prove for abstract blowup squares. This follows by the proposition 5.1.5 below. □

Proposition 5.1.5 (Proper excision) **.** [Kh22, theorem 6.1] Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper representable morphism which induces an isomorphism away from a closed substack $\mathcal{Z} \subset \mathcal{Y}$, then the commutative square

$$\begin{array}{ccc} id & \longrightarrow & j_*j^* \\ \downarrow & & \downarrow \\ f_*f^* & \longrightarrow & g_*g^* \end{array}$$

is homotopy cartisian, where $g : f^{-1}(\mathcal{Z}) \to \mathcal{X}$.

Proof. It suffices to prove the case f is projective. Then by localization, apply i^* and j^*. The statement follows from proper base change and smooth base change. □

In [Kh22], Khan shows that, with some slight modification, the motivic homotopy theory for scalloped stacks recovers the equivariant motivic homotopy theory in the sense of [Hoy17]. And theorem 5.1.4 is a generalization of cdh decent in equivariant homotopy K-theory [Hoy20]. It is possible to prove theorem 5.1.4 outside the framework of the formalism of six operations. See remark 10.6 of [Kh22].

References

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